Vol. 8, No. 3, May 2008, pp. 115–124
ISSN 1532-0545 | 08 | 0803 | 0115

INFORMS Transactions on Education

INFORMS

DOI 10.1287/ited.1080.0009

Teaching Nonzero Sum Games Using a Diagrammatic Determination of Equilibria

Evangelos F. Magirou, Dimitrios K. Vassilakis, Ion Androutsopoulos
Department of Informatics, Athens University of Economics and Business, Patission 76, GR-104 34 Athens, Greece
{efm@aueb.gr, dkv@aueb.gr, ion@aueb.gr}

We present a diagrammatic method that allows the determination of all Nash equilibria of $2 \times M$ nonzero sum games, extending thus the well known diagrammatic techniques for $2 \times M$ zero sum and 2×2 nonzero sum games. We show its appropriateness for teaching purposes by analyzing modified versions of the prisoners' dilemma, the battle of the sexes, as well as of the zero sum game of matching pennies. We then use the method to give simple proofs for the existence of Nash equilibria in all $2 \times M$ games, for both the nonzero sum (Nash existence theorem) and the zero sum case (Minimax theorem). We also prove in the same spirit the remarkable general fact that in a nondegenerate $2 \times M$ nonzero sum game there is an odd number of equilibria.

Key words: nonzero sum games; Nash equilibrium; mixed strategy
History: Received: February 14, 2006; accepted: March 12, 2008. This paper was with the authors 1 year and 5 months for 3 revisions.

1. Introduction

The analysis of competitive behavior is an integral part of economics as well as of operations research. The fundamental concepts date back at least to Cournot's analysis of oligopoly in 1838 through his solution concept which is now embraced by the well known concept of a Nash equilibrium. This equilibrium concept simply states that for a choice of action among the competing players to be stable it must be an optimal response to the actions chosen by the competitors. This equilibrium requirement is often met by more than one set of actions by the participants in a game. It thus does not suffice by itself to define the "solution" to most competitive situations. Several refinements, some rather recent and some that were awarded Nobel Prizes in economics, have extended the applicability of this concept to important economic situations.

A situation where the equilibrium requirement is sufficient to define a "solution" is in the case of two person zero sum games; the solution, referred to as "equilibrium" is unique and its determination can be formulated as a linear program. Zero sum games are thus relatively uncomplicated and appropriate to be included even in an introductory treatment of operations research.

It is noteworthy that several textbooks (Hillier and Lieberman 2001, Vatter et al. 1978, Owen 1982, and Dixit and Skeath 1999) include, in addition to the linear programming approach, a diagrammatic procedure that determines mixed strategy equilibria in simple zero sum games where one player is limited to just two strategies. This diagrammatic approach complements the advanced linear programming arguments and helps to make the concept of a mixed strategy more tangible.

While zero sum games model pure conflict, more general situations with elements of both common interest and conflict, as in oligopoly, auctions, bargaining and negotiation, require a nonzero sum game formulation. It is in such applications that one requires additional conditions for an equilibrium to be a solution of some sort. Most of these additional requirements are stated in terms of the extended (decision tree) form of the game and are thus beyond the level of detail one usually covers in introductory instruction, where the starting point is the normal (matrix) form of the game. However, finding all equilibrium points in general sum games is important for any more refined analysis. It is also difficult, because the methods to determine equilibria exhibit exponential complexity and furthermore, when it comes to the educational aspect, there is no analogue of the $2 \times M$ zero sum game diagrammatic method for general sum ones.

In teaching a games and negotiations class in a Master's program for business mathematics in Greece (jointly organized by the University of Athens and the Athens University of Economics and Business), we employ such an extension of the $2 \times M$ diagrammatic method to exhibit all equilibria of nonzero sum games. We then use it to illustrate for these games theoretical results such as Nash's existence theorem, the fact that the number of nondegenerate Nash equilibria is odd, and finally the zero sum Minimax theorem without linear programming. Our students had no difficulty with this material—most of their difficulties being in modelling competitive situations with trees and the related information structures—and their main complaint was that we did not present enough economic applications (we used relatively few motivating examples from Cabral 2000). About half of the students in this program have considerable mathematical sophistication, being mathematics or engineering graduates, but the other half are economics and business graduates. Neither the first nor the second group had any difficulty with this material.

In this paper we present the method, its application to some educational examples, and its use in elementary proofs for some theoretical results. The method first appeared in Androutsopoulos et al. (2005), where it was used to analyze spam mail traffic.

2. Diagrammatic Solutions for Constant Sum and 2×2 Bimatrix Games

The Nash-Cournot equilibrium condition for a choice of strategies is that each player's chosen action is an optimal reaction to those used by the others. It is evident that this requirement is just one of many desired features of a collective strategy selection before it is called a solution of some sort. Even so, it is challenging to determine all actions satisfying this requirement.

In two player zero sum game where one player has exactly two strategies while the other has an arbitrary number, referred to as $2 \times M$ *zero sum games*, an indirect but easy to implement determination of the equilibrium is as follows (Hillier and Lieberman 2001, Dixit and Skeath 1999): Denote by a_{ij} the payoff to Player I when he uses strategy i while his opponent uses j. If Player I adopts a mixed strategy that uses the first strategy with frequency p while his opponent uses pure strategy j, then Player I's expected outcome is $pa_{1j} + (1-p)a_{2j}$ and his worst possible outcome is the minimum over the opponent's actions. Player can thus guarantee a payoff of $\max_p\{\min_j[pa_{1j} + (1-p)a_{2j}]\}$, a quantity usually denoted by V_{I}. The diagrammatic determination of V_{I} is done by examining the straight lines $[p(a_{1j} - a_{2j}) + a_{2j}]$ and taking their minimum over j. This is a convex function in p with a unique maximum attained at frequency say p^*.

Similarly Player II can guarantee a loss bounded by $V_{\mathrm{II}} = \min_q\{\max_i[\sum_j q_j a_{ij}]\}$ with q being a distribution vector q_j, $j = 1, \ldots, M$. The quantity V_{II} can *not* be conveniently presented diagrammatically, since it is parameterized by two or more parameters q_j. However, in view of the duality of the definitions of the guaranteed levels (the Minimax theorem), the two levels are equal, i.e., $V_{\mathrm{I}} = V_{\mathrm{II}}$. Thus, the mixing parameter p^* obtained in Player I's determination of his guaranteed payoff is actually I's equilibrium strategy. Player II's equilibrium can also be determined by examining his strategies that are effective when p^* is used, which in $2 \times M$ games are usually two. Thus, the justification of this method rests on the duality results of linear programming, an uncomfortable situation when presenting the material to non-OR students.

One can also easily determine the mixed strategy equilibria in 2×2 *nonzero sum games*: for a mixed strategy parameter p of Player I to be part of an equilibrium in a 2×2 game, it must make his opponent indifferent among his two strategies, and vice versa. This requirement can be verified directly: denote by a_{ij} and b_{ij} the payoffs to Players I and II respectively when I uses strategy i and II strategy j. Then, for mixing parameter p of Player I to be part of an equilibrium, it must make Player II indifferent among his choices, namely $b_{11}p + (1-p)b_{21} = b_{12}p + (1-p)b_{22}$; similarly for II's mixing parameter q the requirement is $a_{11}q + (1-q)a_{12} = a_{21}q + (1-q)a_{22}$. Finding whether such mixing parameters exist can be done diagrammatically as in Dixit and Skeath (1999). Extending this procedure to $2 \times M$ games is the objective of this paper.

These diagrammatic methods for $2 \times M$ zero sum and 2×2 nonzero sum games are often used in presenting the well known games of the Prisoners' Dilemma, the Battle of the Sexes, and Matching Pennies (Dixit and Skeath 1999). We present these three games and their usual treatment and note that even a slight, reasonable change in their formulation makes a diagrammatic presentation impossible.

2.1. Example 1: Prisoners' Dilemma

In a prisoners' dilemma situation set as a two company pricing game, there are two firms, each of which produces one product. If firms set prices π_1 and π_2, there will be profits $K_1 = K_1(\pi_1, \pi_2)$ and $K_2 = K_2(\pi_1, \pi_2)$, respectively. Assume that the firms consider only two distinct price levels, 1 and 3, and let the profits resulting for each firm be as in Table 1. Underlines show reactions, and double underlines show pure strategy equilibria.

The pure strategy Nash equilibrium is for both firms to set a price of 1, with resulting profit of 5 units

Table 1 **A Typical Prisoners' Dilemma Game**

Prices	1	3
1	$(\underline{5}, \underline{5})$	$(\underline{9}, 3)$
3	$(3, \underline{9})$	$(8, 8)$

Figure 1 Prisoners' Dilemma: Strategy Outcomes as a Function of Opponent's Strategy Choice

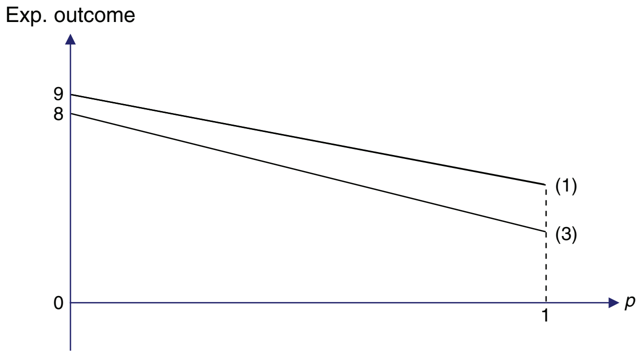


Table 2 A Modified Prisoners' Dilemma Game

Prices	1	3	4
1	(5, 5)	(9, 3)	(9.5, 6)
3	(3, 9)	(8, 8)	(10, 7)

each, while the Pareto efficient outcome that results when a price of 3 is used is not stable, as both players have a motive to deviate. No nontrivial mixing of strategies ($0 < p < 1$) can lead to an equilibrium, since there is strict domination in the strategies and no mixing can make an opponent indifferent among his strategies. Figure 1 demonstrates this domination by plotting the outcome of the two strategies of a player, parameterized on his opponent's mixing parameter.

One can meaningfully extend the example by assuming that Firm II has an extra option to use a price level of 4, for example. This price level is quite beneficial when the opponent's price is 1 but harmful when the rival firm's price level is 3. The profits are assumed as in Table 2. No pure strategy Nash equilibrium exists, but the 2×2 diagrammatic method can be used if we remove Player II's dominated price strategy "3"—provided one is justified in removing dominated strategies. This is indeed so, as will be shown by using the extension of the 2×2 method in §3.

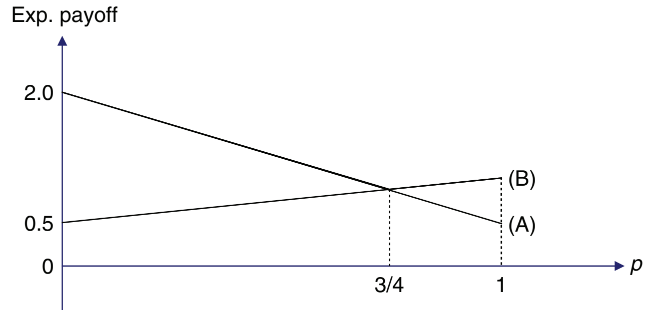
2.2. Example 2: Battle of Sexes

In a battle of sexes game, two firms are considering launching two complementary products, either A or B. Table 3 describes the resulting profits. A firm that develops Product A alone gets a handsome profit, while a firm that develops Product B alone gets a reasonable profit. When both firms launch the same product, Firm I has no profit while Firm II achieves a

Table 3 A Battle of Sexes Game

Products	A	B
A	(0, 0.5)	(2, 1)
B	(1, 2)	(0, 0.5)

Figure 2 Battle of Sexes: Payoffs to II as a Function of Firm I's Strategy Choice



slightly positive profit, perhaps due to its better distribution network.

Two pure strategy Nash equilibria are shown by double underlines: Firm I launches Product A and Firm II launches its complement B, or the reverse. In addition to these equilibria, the players can use a mixing such as to render their opponent indifferent between his two strategies. This is observed as "crossing" in the opponent's strategies payoff in Figures 2 and 3. In the mixed strategy equilibrium Firm I mixes the two products with probabilities 3/4 and 1/4, while Firm II mixes with probabilities 2/3 and 1/3, resulting in profits of 2/3 and 7/8 for I and II, respectively. These are all the available equilibria and their number is indeed odd, in agreement with the result on the counting of equilibria (Lemke and Howson 1964 or Shapley 1974).

Assume now that Firm II has two additional options: to launch Products A′ or B′, which are substitutes of Products A and B respectively. We assume that Product A′ is sold at a higher price than Product A. Therefore, when Firm I launches Product A and Firm II launches Product A′, a substantial portion of demand goes to Firm I. However, when Firm I launches Product B and Firm II launches Product A′, the demand remains unaffected for both firms (since Products B and A′ are complements), while the profit of Firm II increases. We make similar assumptions when Firm II launches product B′ (complement of

Figure 3 Battle of Sexes: Payoffs to Firm I as a Function of Firm II's Strategy Choice

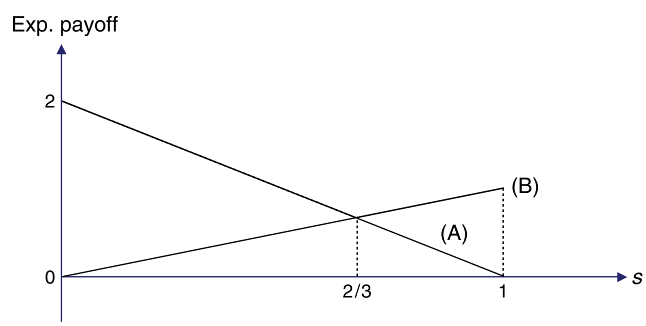


Table 4 A Modified Battle of Sexes Game

Products	A	A′	B	B′
A	(0, 0.5)	(1.5, 0)	(2, 1)	(2, 1.5)
B	(1, 2)	(1, 2.5)	(0, 0.5)	(2.5, 0)

Table 5 A Matching Pennies Game

	Heads	Tails
Heads	(1, −1)	(−1, 1)
Tails	(−1, 1)	(1, −1)

Table 6 A Modified Matching Pennies Game

	Heads	Tails	Stop
Heads	(1, −1)	(−1, 1)	(ε, ε)
Tails	(−1, 1)	(1, −1)	(ε, ε)

Product A and substitute of Product B) and obtain a normal form representation of this game as shown in Table 4. No pure strategy equilibrium exists and the 2×2 method can not be used to determine the mixed strategies in this 2×4 game.

2.3. Example 3: Matching Pennies

In the traditional matching pennies game, each of two players announces Heads or Tails independently of each other. If the announcements match, Player I wins one monetary unit from Player II, and conversely if the calls do not match. The game matrix is shown in Table 5.

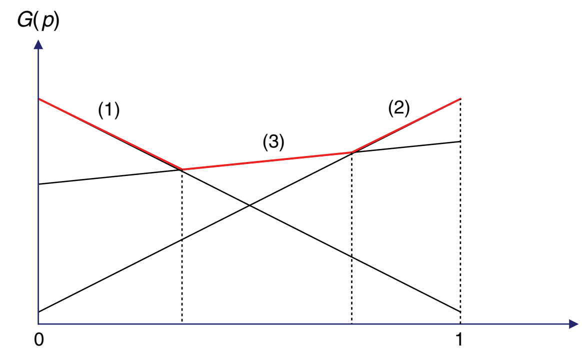
The 2×2 nonzero or zero sum analysis shows that both players should equally mix their calls at 50–50. Assume now that Player II has a third strategy, called "Stop," which stops the game and results to a benefit of ε units to each player, with $0 < \varepsilon < 1$. The payoff matrix of this 2×3 nonzero sum game is shown in Table 6.

It will be instructive to see whether the introduction of the Stop strategy (which makes the situation a nonzero sum one) alters its strategic features.

3. A Diagrammatic Procedure for $2 \times M$ Nonzero Sum Games

We now extend the methods for 2×2 *nonzero* and $2 \times M$ *zero sum games* to $2 \times M$ *nonzero sum games*. The idea of the extension is as follows: first, determine diagrammatically the "reaction curve" of Player II to Player I's mixing parameter p. To do this, plot the outcome of Player II's payoff resulting from his j-th strategy against p. The "reaction" function, $G(p)$, is the maximum of these lines and is shown in red in Figure 4). Each point on this curve is a candidate for

Figure 4 Player II's "Reaction Curve" (Red Envelope)



Nash equilibrium, and, in effect, will be an equilibrium provided Player II can use his "reaction" strategies to make his opponent indifferent between his own two strategies, and hence unwilling to change the mixing parameter that led to the particular point on the maximum curve.

In particular, when Player I mixes with a nontrivial p, $0 < p < 1$, assume that Player II reacts with any of strategies $j_1, j_2, \ldots, j_k$, while the other strategies give a lower payoff. If $k > 1$, we are at a corner of the diagram. If the j_ms ($m = 1, \ldots, k$) can be mixed in a way that makes p an optimal response to them, the mixing parameter p is an equilibrium. For p to be an optimal response there must be a mix of the j_ms that makes Player I indifferent among his two pure strategies. This is not possible if for all of Player II's choices j_m, the payoffs resulting from a particular strategy of Player I dominate those of the other strategy. But if no such domination exists, it can be shown (by Farkas' lemma) that a proper mix can always be found that makes Player I indifferent in his strategies.

If $k = 1$, we are at a linear section of the diagram, corresponding to say strategy j^* of Player II. For these points to form an equilibrium, Player I must be indifferent between his two strategies when Player II uses j^*. Thus, either the entire linear segment is composed of equilibria (if $a_{1j*} = a_{2j*}$), or none of its points is one.

The procedure for mixed equilibria determination is as follows:

(a) Draw the outcome to Player II of each of his M strategies parameterized on his opponent's mixing parameter p, and mark the maximum of these lines—the "Reaction Curve."

(b) For each point on the "reaction curve":

(b.1) For a corner formed by two strategies, say j_1 and j_2, check whether some mixing of j_1 and j_2 can make Player I indifferent between his two strategies. This is possible provided there is no row domination with the game restricted to the j_1 and j_2 columns.

(b.2) More generally, for corners formed by intersections of several strategies $j_1, j_2, \ldots, j_k$ for Player II, check again if there is a mix of any of them making Player I indifferent among his two strategies. This is possible if in the game restricted to these columns there is no row domination for Player I.

(b.3) For a linear segment corresponding to strategy j^* of Player II, compare Player I's payoffs a_{1j*}, a_{2j*}. If they are equal, all points on the segment are equilibrium points; otherwise there are no equilibria on the segment.

We provide an algebraic formulation of this method and some additional numerical examples in Appendix I.

The above procedure is a two step one, the first step being to obtain the "reaction" diagram and the second to examine each point on the diagram to see if it can lead to an equilibrium. It is possible to include information for both players in a single diagram which is more complicated than the original one but which shows immediately the existence of equilibria. This extension consists of adding (in the same diagram) the payoff that would result to Player I, had Player II used his optimal policy corresponding to the choice of the mixing parameter p. The resulting diagram also shows what would happen if Player I were to play first and Player II were to react knowing I's choice of p, namely the Stackelberg solution concept. The maximum (supremum) of this diagram would be the Stackelberg solution with Player I as the leader. Even in simultaneous play (where Nash equilibria are sought), one can still use this *Stackelberg type diagram* to identify Nash equilibria with the following rules:

- **Linear parts:** A horizontal linear section consists of equilibria; any other linear section does not contain equilibria.
- **Discontinuities:** If on a discontinuity the slopes of the adjacent strategies are of the same sign, there is domination and the discontinuity is not an equilibrium. If the slopes have different signs, an equilibrium exists. The value to Player I resulting from this equilibrium is obtained by the intersection of the two adjacent lines[1] as shown in the Figures 5 and 6, while the Player II's value is the value at the corresponding corner of the "reaction" diagram.

Figure 5 depicts this combined "Stackelberg" diagram, where the green lines plot Player I's expected payoff. Among discontinuities A and B, only Product A is a mixed Nash equilibrium: Player II mixes strategies (1) and (3) and Player I mixes his two strategies. The nondomination criterion fails in the case of discontinuity B.

[1] Call the consecutive strategies j and k. At the intersection of their graphs at mixing say p it must be $a_{1j}p + a_{2j}(1-p) = a_{1k}p + a_{2k}(1-p)$. If the mixing parameter used by Player II in the equilibrium is s, it must be that $a_{1j}s + a_{1k}(1-s) = a_{2j}s + a_{2k}(1-s)$. Multiplying the first equation by s, then by $(1-s)$, reversing and adding shows that all these expressions have a common value, which is the value of Player I.

Figure 5 Player II's "Reaction Curve" (Red) and Player I's Expected Payoff (Green)

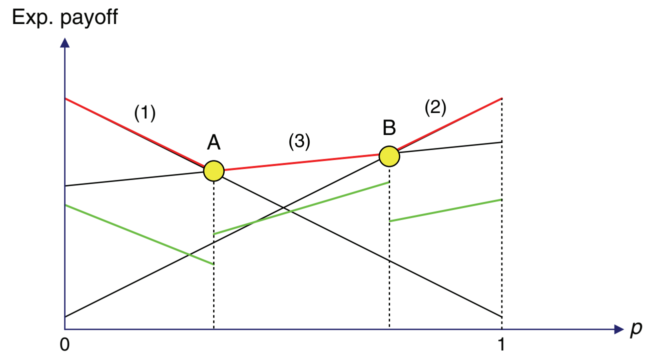


Figure 6 Stackelberg Diagram Approach: Some Cases

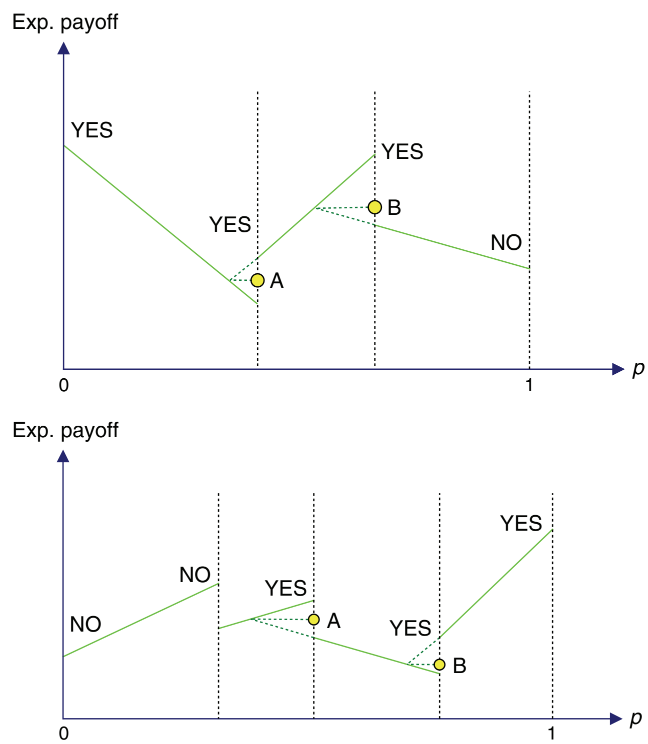


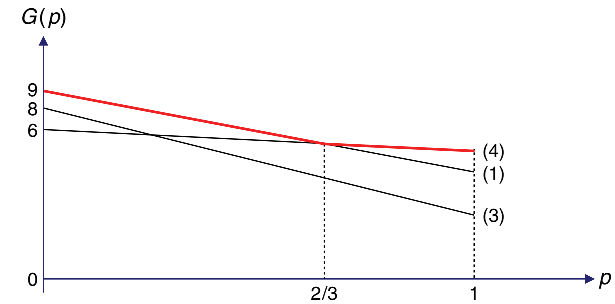
Figure 6 illustrates the determination of Nash equilibria using the combined "Stackelberg" diagram. The intersection of two consecutive lines of slopes of different signs (e.g., points A, B) determines the value of the game.

We illustrate the above procedure to the games introduced in the previous section.

3.1. Example 1: Prisoners' Dilemma

To analyze the modified prisoners' dilemma game shown in Table 2 we construct the corresponding "reaction" diagram (Figure 7). There is only one corner, namely at $p = 2/3$, and Player II's active strategies are 1 and 4. The game restricted to these columns does not show domination and if Player II uses price levels 1 and 4 with probabilities 1/5 and 4/5, Player I

Figure 7 Modified Prisoners' Dilemma: Best Response of II as a Function of I's Strategy Choice



is indifferent among his two strategies and thus has no motive to deviate from the probability 2/3 derived above. This leads to a profit level of 43/5 for the first firm—quite an improvement from the level of 5—and a profit level of 19/3 for the second firm, again an improvement with respect to the prisoners' dilemma result. Thus, the introduction of the extra price level of 4 is beneficial to both players. This is because the new level 4 is a better reaction to 1 than the jointly inferior price level of 1.

The 2×2 method could be used in this example, because Player II's column labeled 3 does not influence the maximum envelope—being dominated by column labeled 1—and thus the 2×3 game reduces to a 2×2 one.

3.2. Example 2: Battle of Sexes

Consider the modified battle of sexes game between two firms launching new products, as shown in Table 4. The "reaction" diagram for Firm II is given in Figure 8. The linear parts do not present equilibria, so we examine the two corners denoted by mixing parameters p_1 and p_2. At p_1 an equilibrium is obtained where Firm I mixes Products A and B with probabilities 1/2 and 1/2 and Firm II mixes Product A and its more expensive substitute A′ with probabilities 1/3 and 2/3. At p_2 there is domination in Firm I's strategies and, hence, it does not present an equilibrium.

Figure 8 Modified Battle of Sexes: Best Response of Firm II as a Function of Firm I's Strategy Choice

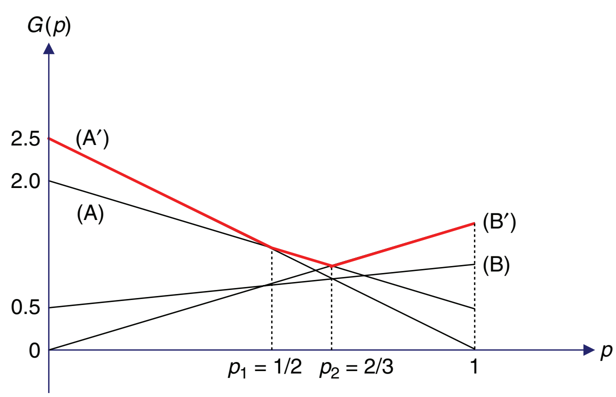
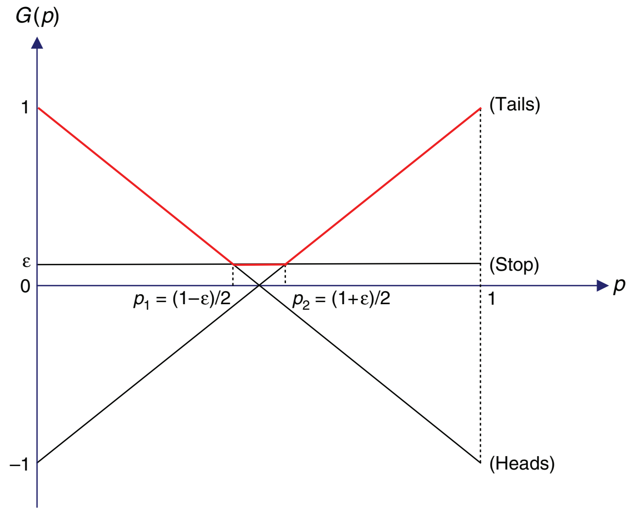


Figure 9 Modified Matching Pennies: Best Response of Player II as a Function of Player I's Strategy Choice



The expected payoffs resulting from the equilibrium at p_1 are 1 and 1.25 to Firms I and II respectively, which is better for both players with respect to the mixed strategy equilibrium in the original game.

3.3. Example 3: Matching Pennies

In the extension of the matching pennies game (Table 6), examining Player II's "reaction" diagram (Figure 9) we observe that there exists an infinity of equilibria in the middle linear section, each resulting to a value of ε for each player. The strategies are interesting: Player II always uses his Stop strategy, while Player I mixes with probability p, $(1-\varepsilon)/2 \le p \le (1+\varepsilon)/2$, which for small ε's is essentially equal to 1/2, the mixing parameter in the zero sum version.

4. Some Important Theorems in the $2 \times M$ Case

The diagrammatic analysis can be used to illustrate for $2 \times M$ games some advanced results. In this section we give informal arguments, while formal arguments are provided in Appendix II.

Starting with *Nash's existence theorem*, we show that any $2 \times M$ game has at least one equilibrium in mixed or pure strategies. If there is an equilibrium in a linear segment, we do not need to look any further. Otherwise, we examine the corners of Player II's "reaction" diagram. If none is an equilibrium, then there must be domination with respect to player I at every corner, and since each of Player II's strategies is involved in two consecutive corners, the domination involves always the same superior strategy for Player I. This means that one of the two rows of Player I dominates the other. But then there exists an equilibrium in pure strategies.

The *Minimax theorem* for zero sum $2 \times M$ games can also be proven. Consider the minimum point of the "reaction" diagram for a zero sum game and assume that it is at an interior corner. At this minimum point the payoffs to Player I can not exhibit dominance, since at an interior minimum one slope is negative and the other is positive and Player I's payoffs are the negatives of Player II's. Hence, the minimum is an acceptable equilibrium, but at any other corner the slopes are of the same sign and thus do not lead to equilibria. A similar analysis holds for a minimum on a linear segment.

We can finally show that *the number of Nash equilibria in a* $2 \times M$ *bimatrix non degenerate game is odd*. With no loss of generality renumber the strategies of Player II so that the corners correspond to consecutive strategies. At each corner of the diagram, if it corresponds to an equilibrium there is a change in the preference of Player I's two strategies as the opponent's strategy changes and, if not, the preference is conserved. If the leftmost point ($p = 0$) is not an equilibrium, an even sequence of equilibria will change the preference of the two strategies and render the rightmost point $p = 1$ an equilibrium, leading thus to an odd number of equilibria. A similar argument holds if the leftmost point is an equilibrium.

5. Conclusions

We showed how a diagrammatic method, in the spirit of the well known methods for $2 \times M$ zero sum and 2×2 bimatrix games, can be applied to $2 \times M$ bimatrix games to determine equilibria. We used it to find equilibria in modified versions of the prisoners' dilemma, battle of sexes and matching pennies games. We finally provided an application of the method to show for the $2 \times M$ case some important theorems of game theory.

Although the procedure presented is simple, it is useful as a teaching tool. In principle, it can be used to deal with $3 \times M$ cases, although no simple domination criterion is applicable to check whether corners lead to equilibria or not. Thus, it is not clear whether a Nash existence proof can be given in this way. It is also not clear whether a similar study of the "reaction curve" can lead to more transparent presentations of the minimax theorem or the property of odd number of equilibria.

Acknowledgments

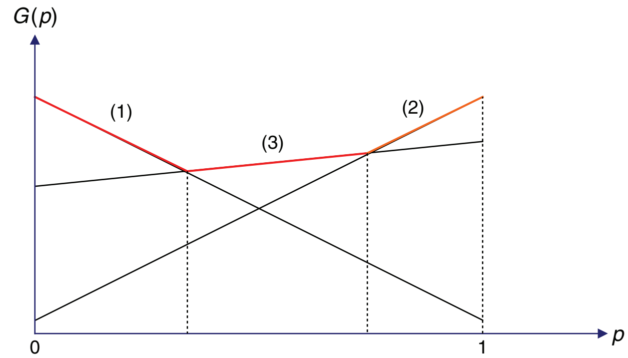
The authors are indebted to the Editor and the anonymous referees for their suggestions that literally transformed the paper. In particular, the use of Stackelberg type diagrams was suggested by one of the referees. The usual caveat applies.

Appendix I

Algebraic Presentation of the Diagrammatic Method

Consider a $2 \times M$ bimatrix game $(A, B) = (a_{ij}, b_{ij})$, $i = 1, 2$ and $j = 1, 2, \ldots, M$. Player I (selecting rows) has two pure strategies, while Player II (selecting columns) has M ($M \geq 2$) pure strategies. Matrices A and B respectively describe Player I and Player II payoffs. Without loss of generality we assume that no row or column is dominated.

Figure A.1 **Value Function** $G(p)$ **for Player II as a Function of Player I's Strategy Choice**



Assume that Player I adopts a mixed strategy $(p, 1 - p)$ with $0 \leq p \leq 1$. If Player II knows p, he maximizes his payoff over his strategies and his optimal payoff is given by the following piecewise linear convex function of p:

$$G(p) = \max_{j=1,2,\ldots,M} \{b_{1j} p + b_{2j}(1 - p)\}$$
$$= \max_{j=1,2,\ldots,M} \{(b_{1j} - b_{2j})p + b_{2j}\} \tag{A1}$$

Player II's optimal "reaction" to I's choice of p is the best response strategy set $J(p)$ with:

$$J(p) = \{j \mid (b_{1j} - b_{2j})p + b_{2j} = G(p)\}. \tag{A2}$$

The pure strategies in $J(p)$ maximize Player II's expected payoff given Player I's mixed strategy $(p, 1 - p)$. Hence, the best response of Player II to Player I's choice of p is any mixed strategy on the strategies in $J(p)$. A bimatrix game is usually called *nondegenerate* if the number of pure best responses to a mixed strategy never exceeds the size of its support (i.e., the strategies used in the mixed strategy); otherwise the game is called *degenerate* (Von Stengel 2002).

Figure A.1 demonstrates the function $G(p)$. On the linear parts of $G(p)$ the best response set $J(p)$ consists of a single pure strategy, while at the corners the response set $J(p)$ consists of as many pure strategies cross at a specific corner, typically two.

An *equilibrium* is a pair $(\vec{p}^*, \vec{s}^*)$ of mixed strategies of the two players where no player has an incentive to deviate unilaterally. In particular, an equilibrium is a pair $(\vec{p}^*, \vec{s}^*)$, $\vec{p}^* \in R^2$ and $\vec{s}^* \in R^M$, corresponding to discrete distributions and such that $U_{\text{I}}(\vec{p}^*, \vec{s}^*) \geq U_{\text{I}}(\vec{p}, \vec{s}^*)$ and $U_{\text{II}}(\vec{p}^*, \vec{s}^*) \geq U_{\text{II}}(\vec{p}^*, \vec{s})$ for every $\vec{p}$ and $\vec{s}$, where U_{I} and U_{II} denote the expected payoffs to Players I and II respectively.

Consider Player I's mixed strategy $\vec{p} = (p, 1 - p)$, where $|J(p)| = k$ and $J(p) = \{j_1, j_2, \ldots, j_k\}$, meaning that Player II may adopt a mixed strategy-distribution on the k pure strategies contained in $J(p)$. We obtain an equilibrium if and only if a mixed strategy[2] $\vec{s}=(s_{j_1}, s_{j_2}, \ldots, s_{j_k})$ for Player II satisfies:

$$\sum_{j\in J(p)} a_{1j}s_j = \sum_{j\in J(p)} a_{2j}s_j$$
$$\sum_{j\in J(p)} s_j = 1 \qquad \text{(A3)}$$
$$s_j \geq 0 \quad \forall j \in J(p)$$

If this system is feasible, Player I is indifferent between his two pure strategies and has no incentive to deviate from his mixed strategy $\vec{p}$. Player II is also indifferent between his strategies $j_1, j_2, \ldots, j_k$, since they are all optimal reactions to p by the definition of $J(p)$.

The case $|J(p)|=1$ ($J(p)=\{j^*\}$) corresponds to the linear parts of $G(p)$. We obtain an equilibrium if and only if $a_{1j*}=a_{2j*}$, where $s_{j*}=1$. This leads to a continuum of equilibria corresponding to Player II's pure strategy j^* and any mixture $(p, 1-p)$ with $p_i \leq p \leq p_{i+1}$ of Player I. On any internal point of this line segment, the number of Player I's pure best responses to Player II's strategy j^* exceeds the size of its support (which is one, since j^* is a pure strategy) and the game is degenerate. A continuum of equilibria may exist only in degenerate bimatrix games.

A case with $|J(p)|=k$, $k\geq 2$ corresponds to a corner of function G, being the intersection of Player II's pure strategies $j_1, j_2, \ldots, j_k$. It can be proven using Farkas' Lemma that an equilibrium mixed strategy $\vec{s}=(s_{j_1}, s_{j_2}, \ldots, s_{j_k})$ (or a family of equilibria) exists, if and only if Player I's payoff Matrix A restricted to columns $j_i \in J(p)$ $i=1,\ldots,k$ demonstrates no domination in the rows.[3] This result is important in proving the general case of Nash's existence theorem for $2\times M$ games. The nondomination criterion is easy to establish in case $|J(p)|=2$ ($J(p)=\{j_1, j_2\}$): in an equilibrium, the strategy $\vec{s}=(s, 1-s)$ used by Player II should leave Player I indifferent between his two pure strategies, which means that $a_{1j_1}s + a_{1j_2}(1-s) = a_{2j_1}s + a_{2j_2}(1-s)$. If $a_{2j_1}=a_{1j_1}$ and $a_{2j_2}=a_{1j_2}$, this equality holds for any value of s and, hence, we obtain a continuum of equilibria at p. Otherwise, it has a unique solution:

$$s=\left(1-\frac{a_{2j_1}-a_{1j_1}}{a_{2j_2}-a_{1j_2}}\right)^{-1} \qquad \text{(A4)}$$

and an equilibrium exists if $0\leq s\leq 1$, or equivalently:

$$(a_{2j_1}-a_{1j_1})(a_{2j_2}-a_{1j_2}) \leq 0 \qquad \text{(A5)}$$

with equality holding when $s=0$ or $s=1$. This states that the A matrix game restricted in columns j_1 and j_2 is not dominated in the rows.

Notice that when $|J(p)|\geq 3$, existence of equilibrium implies that the game is degenerate. This is because the size of support for Player I's mixed strategy $\vec{p}=(p, 1-p)$ (equals two), is less than the number of Player II's pure best responses to $\vec{p}$.

[2] For simplicity, we omit zero valued elements and we let vector $\vec{s}$ contain only elements corresponding to those strategies intersecting at p.

[3] The proof is available from the authors upon request.

Table A.1 **Strategic Form of the Example Game (Three Mixed Equilibria)**

	II			
I	1	2	3	4
1	(3, −2)	(1, 2)	(4, 6)	(2, 8)
2	(1, 12)	(5, 10)	(2, 4)	(3, −4)

Examples

Example 1. Consider the following 2×4 bimatrix game, where Matrix A describes Player I's payoffs and Matrix B Player II's payoffs. Table A.1 demonstrates the strategic (normal) form of this game.

$$A=\begin{bmatrix}3 & 1 & 4 & 2\\ 1 & 5 & 2 & 3\end{bmatrix} \quad B=\begin{bmatrix}-2 & 2 & 6 & 8\\ 12 & 10 & 4 & -4\end{bmatrix}$$

There is no dominance in this game and no pure strategy equilibrium exists. If Player I adopts a mixed strategy (p, $1-p$), $0\leq p\leq 1$, then Player II's reaction function $G(p)$ is shown diagrammatically in Figure A.2.

The three corners of $G(p)$ are determined by solving equations:

(1)–(2): $-14p+12=-8p+10 \;\Rightarrow\; p_1=1/3$

(2)–(3): $-8p+10=2p+4 \;\Rightarrow\; p_2=3/5$

(3)–(4): $2p+4=12p-4 \;\Rightarrow\; p_3=4/5$

We obtain corners at $p_1=1/3$, the intersection of pure strategies 1 and 2, at $p_2=3/5$, the intersection of strategies 2 and 3, and at $p_3=4/5$, the intersection of strategies 3 and 4.

There is no equilibrium on any linear segment of $G(p)$ since $a_{1j}\neq a_{2j}$, $j=1,2,3,4$. However, the three corners are candidates for Nash equilibria. At $p_1=1/3$ Player II should mix his pure strategies (1) and (2) in a way that Player I is indifferent between his two pure strategies, namely: $3s+1-s=s+5(1-s)\Rightarrow s=2/3$, which is a unique solution, so this is an equilibrium case. Similarly, at $p_2=3/5$, $s+4(1-s)=5s+2(1-s)\Rightarrow s=1/3$.

Finally, at $p_3=4/5$: $4s+2(1-s)=2s+3(1-s)\Rightarrow s=1/3$.

Figure A.2 **Numerical Example, Function** $G(p)$

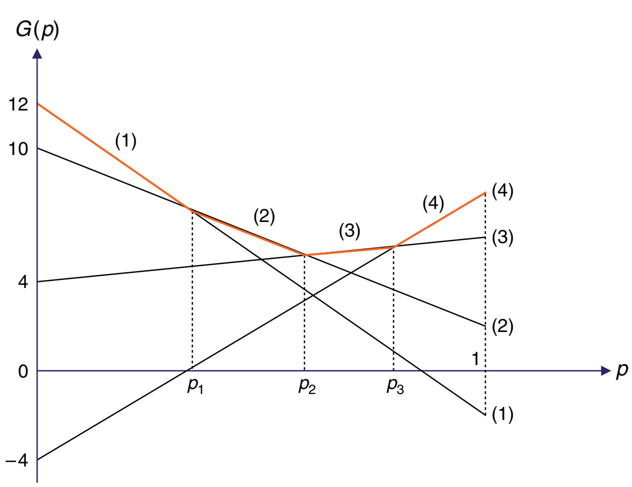


Table A.2 Strategic Form of the Example Game (Only One Mixed Equilibrium)

	II			
I	1	2	3	4
1	(3, −2)	(5, 2)	(4, 6)	(2, 8)
2	(1, 12)	(1, 10)	(2, 4)	(3, −4)

Table A.3 Strategic Form of the Example Game (One Pure Equilibrium Plus Two Mixed

	II			
I	1	2	3	4
1	(1, −2)	(1, 2)	(4, 6)	(2, 8)
2	(**3**, **12**)	(5, 10)	(2, 4)	(3, −4)

We obtain thus three Nash equilibria corresponding to the mixed strategies:

I: $(1/3, 2/3)$ II: $(2/3, 1/3, 0, 0)$

I: $(3/5, 2/5)$ II: $(0, 1/3, 2/3, 0)$

I: $(4/5, 1/5)$ II: $(0, 0, 1/3, 2/3)$

Example 2. It is the structure of the Matrix A that determines which of the corners of $G(p)$ are equilibria. Thus, a mere interchange of the second column elements in Matrix A (see Table A.2) leads to a single equilibrium: although $G(p)$ has the exact same corners as in the previous example, the change in Player I's payoff destroys the equilibrium nature of the first two equilibria, and just the third equilibrium remains.

Example 3. Flipping the first column elements in matrix A (Table A.3), gives rise to a pure Nash equilibrium, i.e., (3, 12), as well as the last two mixed equilibria of Example 1.

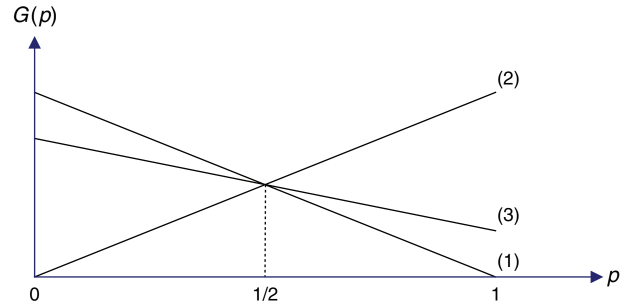
Example 4. The case of a degenerate game is also interesting, since it may give rise to a whole new family of equilibria. An example is the game in Table A.4, with the $G(p)$ graph in Figure A.3, where three pure strategies of Player II intersect.

The family of equilibria is generated by any distribution (s_1, s_2, s_3) of Player II satisfying $3s_1 + s_2 + 2(1 - s_1 - s_2) = s_1 + 2s_2 + 3(1 - s_1 - s_2)$. Thus, the equilibrium calls for Player II to use mixed strategies $(1/3, k, 2/3 - k)$ (where $0 \leq k \leq 2/3$), while Player I adopts $(1/2, 1/2)$. Correspondingly, there is a continuum of equilibrium values: the value for Player I is any number between 5/3 and 7/3, while Player II's value is 2 for all equilibria.

Table A.4 An Example Game Where a Corner Corresponds to Three Strategies

	II		
I	1	2	3
1	(3, 0)	(1, 4)	(2, 1)
2	(1, 4)	(2, 0)	(3, 3)

Figure A.3 Three Strategies Interesting at p



Appendix II

Starting with Nash's theorem, we will show that any $2 \times M$ game has at least one equilibrium in mixed or pure strategies. Assuming that no pure strategy equilibrium exists, consider the corners of $G(p)$ which are candidates for mixed strategy equilibria. Without loss of generality assume that Player II's strategies appear consecutively in the corners of G and that all corners consist of two strategies. Then, if the first corner is not an equilibrium, the game restricted to the first two columns must show strict domination. Without loss of generality we assume that $a_{11} < a_{21}$ and $a_{12} < a_{22}$ hold. In order for the second corner not to be an equilibrium, we must also have domination, namely $a_{12} < a_{22}$ and $a_{13} < a_{23}$—the reverse inequalities are not a possibility in view of the inequalities at the previous corner. Continuing for all corners we get that $a_{1k} < a_{2k}$ for all columns k. This shows that a pure equilibrium would exist in row 2, contradicting the assumption about non existence of pure equilibria. The same argument applies to degenerate games as well (where corners may be formed by more than two strategies), since it can be shown by Farkas' lemma that even for such a corner non existence of equilibrium implies row domination.

The Minimax theorem for zero sum games can also be proven, in fact without using the Nash result established earlier. Consider a zero sum game on a $2 \times M$ matrix A (or equivalently our initial bimatrix game with $b_{ij} = -a_{ij}, \forall i, j$), representing Player I's payoff. Assume that no pure strategy equilibrium exists. We will show that an equilibrium (or continuum of equilibria) appears at the min of max or at the max of min estimates in zero sum game calculations.

Assume that the convex function $G(p)$ has a minimum at say p^*. This minimum must be interior, i.e., $0 < p^* < 1$, for otherwise a pure strategy equilibrium would exist. Let columns i and $i+1$ be the ones[4] determining p^* in the definition of $G(p)$. Then p^* corresponds to a Nash equilibrium since the condition $(a_{2,i} - a_{1,i})(a_{2,i+1} - a_{1,i+1}) \leq 0$ is clearly satisfied (as a strict inequality), because the line segment corresponding to the i-th pure strategy should have a non positive slope (i.e., $b_{1,i} - b_{2,i} < 0 \Rightarrow a_{2,i} - a_{1,i} < 0$), while the one corresponding to the $i+1$-th strategy should have a nonnegative slope (i.e., $b_{1,i+1} - b_{2,i+1} > 0 \Rightarrow a_{2,i+1} - a_{1,i+1} > 0$), due to the minimum property of p^*. At any other intersection point, the corresponding segments should

[4] The intersection of three or more strategies at point p^* does not affect the validity of our discussion.

Table A.5 An Example Game with One Mixed Plus Two Pure Equilibria

	II	
I	1	2
1	(1, 3)	($\underline{4}$, $\underline{4}$)
2	($\underline{4}$, $\underline{5}$)	(3, 2)

Table A.6 Destroying the Pure Equilibria

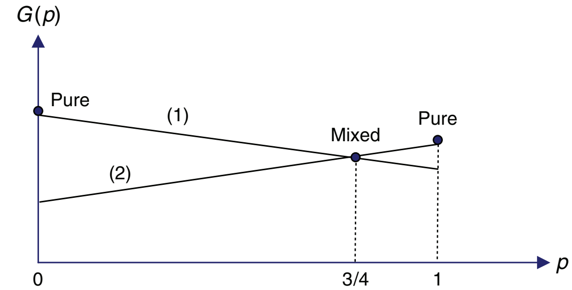
	II	
I	1	2
1	($\underline{4}$, 3)	(3, $\underline{4}$)
2	(1, $\underline{5}$)	($\underline{4}$, 2)

have slopes with the same sign, violating thus inequality $(a_{2j_1} - a_{1j_1})(a_{2j_2} - a_{1j_2}) \leq 0$. The case of non unique p^* can be dealt with similarly.

An alternative proof is possible by using the Nash result. Indeed, the existence of a Nash equilibrium implies that the guarantee levels of the two players are equal, and can be used to calculate the game's value. The guarantee level calculation for Player I determines the maximum of the envelope of the minima and is the one performed by the diagrammatic procedure in case there are only two rows. By Nash's theorem, this maximum is guaranteed to provide an equilibrium for the entire game.

We finally show that the total number of Nash equilibria in a $2 \times M$ bimatrix nondegenerate game is *odd*. Since the game is non degenerate, exactly two Player II's strategies intersect at any corner of $G(p)$. Without loss of generality assume that these strategies appear consecutively in the corners of G. Consider a pure strategy i for Player II and call it *positive* if $a_{1,i} > a_{2,i}$, *negative* otherwise. If the corner $(i, i+1)$ formed by strategies i and $i+1$ is not an equilibrium, then strategies i and $i+1$ have the same property value; otherwise they have opposite values. Consider all corners $(1, 2), (2, 3), \ldots, (M-1, M)$ and assume that there is an even number of mixed strategy equilibria among them. This implies that strategies 1 and M both have the same property and hence exactly one extra pure equilibrium exists, either at strategy 1 or at M. If the corners of $G(p)$ demonstrate an odd number of equilibria, then either none of 1 and M or both participate in a pure equilibrium. Both cases lead to an odd total number of equilibria.

Figure A.4 Three Equilibria



Example 1. The examples in Table A.5 and Table A.6 demonstrate the idea behind the previous proof. In Table A.5, column 1 leads to a pure strategy; the corner at $p = 3/4$ is a mixed equilibrium (see Figure A.4) which forces column 2 to be a pure equilibrium as well, hence an odd number of equilibria.

Interchanging the rows of the Matrix A leads to Table A.6: the corner mixed equilibrium remains, but now column 1 does not lead to a pure equilibrium. Hence neither column 2 can lead to a pure equilibrium and thus the single equilibrium remains.

References

Androutsopoulos, I., E. F. Magirou, D. K. Vassilakis. 2005. A game theoretic model of spam e-mailing. *Proc. IEEE 2nd Conf. Email and Anti-Spam (CEAS 2005), Stanford University.* Institute of Electrical and Electronics Engineers, Washington, D.C.

Cabral, L. 2000. *Introduction to Industrial Organization*. The MIT Press, Cambridge, MA.

Dixit, A., S. Skeath. 1999. *Games of Strategy*, 1st ed. W. W. Norton & Company, New York.

Hillier, F., G. J. Lieberman. 2001. *Introduction to Operations Research*, 7th ed. McGraw Hill, New York.

Lemke, C. E., J. T. Howson, Jr. 1964. Equilibrium points of bimatrix games. *J. Soc. Indust. Appl. Math.* **12** 413–423.

Owen, G. 1982. *Game Theory*, 2nd ed. Academic Press, New York.

Shapley, L. S. 1974. A note on the Lemke-Howson algorithm. M. L. Balinski, ed. *Mathematical Programming Study 1: Pivoting and Extensions*. North-Holland Publishing Company, Amsterdam, 175–189.

Vatter, P., S. Bradley, S. Frey, B. Jackson. 1978. *Quantitative Methods in Management: Text and Cases*. Richard D. Irwin Inc., (n.p.).

Von Stengel, B. 2002. Computing equilibria for two-person games. R. Aumann, S. Hart, eds. *Handbook of Game Theory with Economic Applications*, Vol. 3. North-Holland, Amsterdam, 1723–1759.